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Nonabelian tensor products and Nonabelian homology of groups

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Abstract

The non-abelian tensor product of groups introduced by Brown and Loday is generalized. The homology groups of groups are constructed with coefficients in any group, as the left derived functors of the non-abelian tensor product, which generalize the classical theory of homology of groups. Exact sequences of the non-abelian homology groups and their application to algebraic K-theory of noncommutative local rings are given.

Some functorial properties of the nonabelian tensor product of groups are established. With the use of the nonabelian left-derived functors [7, 8] the homology groups of groups are constructed with coefficients in any group, as the left-derived functors of the non-abelian tensor product, which generalize the classical theory of homology of groups. Exact sequences of the nonabelian homology groups and their application to algebraic K-theory of noncommutative local rings are given. For this we extend the nonabelian tensor product of groups introduced by Brown and Loday [2–4, 5, 10] to arbitrary pairs G, H of groups which act on themselves by conjugation and each of which acts on the other. We do not demand as in [2–4] that the compatibility conditions

$$^{(^gh)}g' = {}^{ghg^{-1}}g', \qquad ^{(^hg)}h' = {}^{hgh^{-1}}h' \tag{1}$$

hold, where $g, g' \in G$, $h, h' \in H$, and ghg^{-1}, hgh^{-1} are elements of the free product $G * H$.

The tensor product $G \otimes H$ is the group generated by the symbols $g \otimes h$ and defined by the relations

$$gg' \otimes h = (^gg' \otimes {}^gh)(g \otimes h),$$

$$g \otimes hh' = (g \otimes h)(^hg \otimes {}^hh'),$$

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$$(g \otimes h)(g' \otimes h') = ({}^{[g,h]}g' \otimes {}^{[g,h]}h')(g \otimes h),$$

$$(g' \otimes h')(g \otimes h) = (g \otimes h)({}^{[h,g]}g' \otimes {}^{[h,g]}h')$$

for all $g, g' \in G$ and $h, h' \in H$, where $[g, h] = ghg^{-1}h^{-1} \in G * H$.

In order to obtain the characterizing universal property of the nonabelian tensor product of groups we need the notion of a crossed pairing. A map $\Phi: G \times H \to L$ where G, H and L are groups, is called a crossed pairing if for all $g, g' \in G$ and $h, h' \in H$,

$$\Phi(gg', h) = \Phi({}^{g}g', {}^{g}h)\Phi(g, h),$$

$$\Phi(g, hh') = \Phi(g, h)\Phi({}^{h}g, {}^{h}h'),$$

$$\Phi(g, h)\Phi(g', h') = \Phi({}^{[g,h]}g', {}^{[g,h]}h')\Phi(g, h),$$

$$\Phi(g', h')\Phi(g, h) = \Phi(g, h)\Phi({}^{[h,g]}g', {}^{[h,g]}h').$$

Definition 1. The nonabelian tensor product of groups G and H is a group A and a crossed pairing $f: G \times H \to A$ such that for every group D and every crossed pairing $f': G \times H \to A$ there exists a unique homomorphism $\alpha: A \to D$ such that $\alpha f = f'$.

It is easy to see the above-mentioned tensor product $G \otimes H$ satisfies this definition and it is unique up to isomorphism. We have the isomorphism $\gamma: G \otimes H \to H \otimes G$, $\gamma(g \otimes h) = (h \otimes g)^{-1}$. It is clear also that if the actions of the pair G, H of groups satisfy the compatibility conditions (1), then we obtain the tensor product of Brown and Loday.

Therefore, if G and H act trivially on each other, then $G \otimes H \approx G^{ab} \otimes H^{ab}$ (the tensor product of the abelianizations of G and H). It follows also that if H is an abelian group which acts trivially on G, then we have the isomorphism of Guin [6] $G \otimes H \approx IG \otimes_G H$, where $IG = \operatorname{Ker} \varepsilon$, $\varepsilon: \boldsymbol{Z}[G] \to \boldsymbol{Z}$, $\varepsilon(\sum_i n_i g_i) = \sum_i n_i$ and this isomorphism is given by $g \otimes h \mapsto (g - 1) \otimes \alpha$ (see [6]).

Suppose $\Theta: G \to A$, $\Phi: H \to B$ are homomorphism of groups, A, B act on each other, and Θ, Φ preserve the actions in the sense that

$$\Phi({}^{g}h) = {}^{\Theta g}(\Phi h), \qquad \Theta({}^{h}g) = {}^{\Phi h}(\Theta g)$$

for all $g \in G, h \in H$. Then there is a unique homomorphism $\Theta \otimes \Phi: G \otimes H \to A \otimes B$ such that $(\Theta \otimes \Phi)(g \otimes h) = \Theta g \otimes \Phi h$ for all $g \in G$, $h \in H$. Further, if Θ, Φ are onto, so also is $\Theta \otimes \Phi$.

In fact, we have

$$\begin{aligned}(\Theta \times \Phi)(gg', h) &= \Theta g \Theta g' \otimes \Phi h = ({}^{\Theta g}\Theta g' \otimes {}^{\Theta g}\Phi h)(\Theta g \otimes \Phi h) \\ &= (\Theta({}^{g}g') \otimes \Phi({}^{g}h))(\Theta g \otimes \Phi h) = (\Theta \times \Phi)({}^{g}g', {}^{g}h)(\Theta \times \Phi)(g, h), \\ (\Theta \times \Phi)(g, hh') &= \Theta g \otimes \Phi h \Phi h' = (\Theta g \otimes \Phi h)({}^{\Phi h}\Theta g \otimes {}^{\Phi h}\Phi h') \\ &= (\Theta g \otimes \Phi h)(\Theta({}^{h}g) \otimes \Phi({}^{h}h')) = (\Theta \times \Phi)(g, h)(\Theta \times \Phi)({}^{h}g, {}^{h}h'),\end{aligned}$$

$$
\begin{aligned}
&(\Theta \times \Phi)(g,h)(\Theta \times \Phi)(g',h')(\Theta \times \Phi)(g,h)^{-1} \\
&\quad = (\Theta g \otimes \Phi h)(\Theta g' \otimes \Phi h')(\Theta g \otimes \Phi h)^{-1} \\
&\quad = {}^{[\Theta g, \Phi h]}\Theta g' \otimes {}^{[\Theta g, \Phi h]}\Phi h' = \Theta({}^{[g,h]}g') \otimes \Phi({}^{[g,h]}h') = (\Theta \times \Phi)({}^{[g,h]}g', {}^{[g,h]}h').
\end{aligned}
$$

Analogously, we can verify the fourth relation of the crossed pairing.

Thus, the map $\Theta \times \Phi : G \times H \to A \otimes B$ is a crossed pairing and $\Theta \otimes \Phi$ is the induced homomorphism.

We now investigate for the nonabelian tensor product the connection with the group of crossed homomorphisms as adjoint functors and the properties of exactness and compatibility with the direct limits of groups.

Theorem 1. (a) *Suppose*

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1$$

is a short exact sequence of groups, D is an arbitrary group which acts on A, B and C, the groups A, B, C also act on D and f, g preserve the actions. Then we have the following exact sequence of groups:

$$D \otimes A \xrightarrow{f'} D \otimes B \xrightarrow{g'} D \otimes C \longrightarrow 1,$$

where $f' = 1 \otimes f$, $g' = 1 \otimes g$.

(b) *Suppose*

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1, \tag{2}$$

$$1 \longrightarrow D \xrightarrow{\phi} E \xrightarrow{\psi} F \longrightarrow 1 \tag{3}$$

are short exact sequences of groups, where A and D, B and E, C and F act on each other, f and ϕ, g and ψ preserve the actions. Then the following sequence of groups:

$$(A \otimes E) \times (B \otimes D) \xrightarrow{\alpha} B \otimes E \xrightarrow{g \otimes \psi} C \otimes F \longrightarrow 1$$

is exact, where α is a map of sets.

Proof. (a) First we shall show that $\operatorname{Im} f'$ is an invariant subgroup of $D \otimes B$. Let $\prod_{i=1}^{k} (d_i \otimes f(a_i))^{n_i} \in \operatorname{Im} f'$. It suffices to prove that

$$(d \otimes b) \prod_{i=1}^{k} (d_i \otimes f(a_i))^{n_i} (d \otimes b)^{-1} \in \operatorname{Im} f'.$$

It is clear that $(d \otimes b) \prod_{i=1}^{k} (d_i \otimes f(a_i))^{n_i} (d \otimes b)^{-1} = (d \otimes b)(d_1 \otimes f(a_1))(d \otimes b)^{-1} \times (d \otimes b) \cdots (d \otimes b)(d_k \otimes f(a_k))(d \otimes b)^{-1} = \prod_{i=1}^{k} ({}^{[d,b]}d_i \otimes {}^{[d,b]}f(a_i))^{n_i} \in \operatorname{Im} f'$, since each ${}^{dbd^{-1}b^{-1}}f(a_i) = {}^{d}(b({}^{d^{-1}}(b^{-1}f(a_i)b))b^{-1})$ is an element of $\operatorname{Im} f = \operatorname{Ker} g$. Therefore, we have diagram

$$\begin{tikzcd} D \otimes A \arrow[r, "f'"] \arrow[d, equal] & D \otimes B \arrow[r, "g'"] \arrow[d, equal] & D \otimes C \arrow[r] & 1 \\ D \otimes A \arrow[r, "f'"] & D \otimes B \arrow[r, "\tau"] & \operatorname{Coker} f' \arrow[r] & 1 \end{tikzcd}$$

where $g'f'(x) = 1$, $x \in D \otimes A$ and the bottom row is exact. Thus, there exists a natural homomorphism β:Coker $f' \to D \otimes C$ such that $\beta\tau = g'$.

Now define a crossed pairing $\bar{\gamma}: D \times C \to \operatorname{Coker} f'$ as follows: $\bar{\gamma}(d, c) = [d \otimes b]$, where $g(b) = c$. It is correctly defined. If $b' = b \cdot f(a)$, then $d \otimes b' = d \otimes b \cdot f(a) = (d \otimes b)({}^b d \otimes {}^b f(a))$. Thus, $(d \otimes b)^{-1}(d \otimes b') = {}^b d \otimes {}^b f(a) \in \operatorname{Im} f'$. Next, $\bar{\gamma}(dd', c) = [dd' \otimes b] = [{}^d d' \otimes {}^d b][d \otimes b] = \bar{\gamma}({}^d d' \otimes {}^d c)\bar{\gamma}(d, c)$, where $g(b) = c$. Further $\bar{\gamma}(d, cc') = [d \otimes bb'] = [d \otimes b][{}^{g(b)}d \otimes {}^b b'] = \bar{\gamma}(d, c)\bar{\gamma}({}^c d, {}^c c')$, $g(b') = c'$. $\bar{\gamma}(d, c)\bar{\gamma}(d', c')\bar{\gamma}(d, c)^{-1} = [d \otimes b][d' \otimes b'][d \otimes b]^{-1} = [{}^{[d,b]}d' \otimes {}^{[d,b]}b'] = \bar{\gamma}({}^{[d,c]}d', {}^{[d,c]}c')$, similarly for the fourth relation.

Therefore, $\bar{\gamma}$ is a crossed pairing and it induces a homomorphism $\gamma: D \otimes C \to$ Coker f' such that $\gamma(d \otimes c) = [d \otimes b]$, $g(b) = c$. It is easy to see that $\beta\gamma$ and $\gamma\beta$ are identity maps.

(b) Define the map

$$\alpha:(A \otimes E) \times (B \otimes D) \to B \otimes E$$

as follows:

$$\left(\prod_i (a_i \otimes e_i), \prod_i (b_i \otimes d_i)\right) \mapsto \prod_i (f(a_i) \otimes e_i) \cdot \prod_i (b_i \otimes \Phi(d_i)).$$

It is clear that $\operatorname{Im} \alpha = \operatorname{Im}(f \otimes 1_E) \cdot \operatorname{Im}(1_B \otimes \phi)$. From the proof of (a) it follows that the groups $\operatorname{Im}(f \otimes 1_E)$, $\operatorname{Im}(1_B \otimes \phi)$ are invariant subgroups of $B \otimes E$. It is easy to see that $\operatorname{Im} \alpha \subseteq \operatorname{Ker}(g \otimes \psi)$ and thus we have a natural homomorphism

$$\pi': B \otimes E/\operatorname{Im} \alpha \to C \otimes F.$$

Next, define a crossed pairing $\bar{p}': C \times F \to B \otimes E/\operatorname{Im} \alpha$ as follows: $\bar{p}'(c, l) = [b \otimes e]$, where $g(b) = c$, $\psi(e) = l$ and show the correctness. If $b' = b \cdot f(a)$, $e' = e\phi(d)$, then $b' \otimes e' = bf(a) \otimes e\phi(d) = ({}^b f(a) \otimes {}^b e^b \phi(d))(b \otimes e\phi(d)) = ({}^b f(a) \otimes {}^b e^b \phi(d)) \times (b \otimes e)({}^e b \otimes {}^e \phi(d))$. Since ${}^e b \otimes {}^e \phi(d)$ is an element of the invariant subgroup $\operatorname{Im}(1_B \otimes \phi)$, it follows that $(b' \otimes e')(b \otimes e)^{-1} \in \operatorname{Im} \alpha$. Next, $\bar{p}(cc', l) = [bb' \otimes e]$, where $g(b) = c$, $g(b') = c'$, $\psi(e) = l$; $\bar{p}'({}^c c', {}^c l) = [{}^b b' \otimes {}^b e]$ since $\psi({}^b e) = {}^{g(b)}\psi(e) = {}^c l$. Thus, $\bar{p}'(cc', l) = \bar{p}'({}^c c', {}^c l)\bar{p}'(c, l)$. Similarly, we can prove the second and the third parts of the crossed pairing. Therefore, the map $\bar{p}'$ is a crossed pairing and it induces a homomorphism $p': C \otimes F \to B \otimes E/\operatorname{Im} \alpha$ such that $p'(c \otimes l) = [b \otimes e]$, where $g(b) = c$, $\psi(e) = l$. It is easy to see that $\pi' p'$ and $p' \pi'$ are identity maps. The proof of Theorem 1 is complete. □

Remark 1. If the sequences (2), (3) coincide and are central extensions, then from Theorem 1(b) follows the result of [2, Proposition 9] and in this case the map α is a homomorphism. Moreover, if the sequence (3) is $1 \to 1 \to D \xrightarrow{1_D} D \to 1$, then we obtain the first part of Theorem 1.

Theorem 2. *Let A and B be groups which act on each other, such that the compatibility conditions* (1) *hold. Let C be a group on which act the groups A and B. Then there is a natural bijection*

$$\tilde{F}(A, Z^1(B, C)) \approx \widetilde{\mathrm{Hom}}(A \otimes B, C),$$

where $\widetilde{\mathrm{Hom}}(A \otimes B, C)$ is the set of homomorphisms satisfying the following conditions:

(1) $g(aa' \otimes b) = g(a'a \otimes b)$,

(2) $g^{a'}(a \otimes b) \equiv g({}^{a'}a \otimes {}^{a'}b) = {}^{a'}g(a \otimes b)$,

(3) $g^{b'}(a \otimes b) \equiv g({}^{b'}a \otimes {}^{b'}b) = {}^{b'}g(a \otimes b)$.

$\tilde{F}(A, Z^1(B, C))$ is the set of maps from A to the set $Z^1(B, C)$ of crossed homomorphisms satisfying the following conditions:

(4) $[f(aa')](b) = {}^{a}[f(a')](b) \cdot [f(a)](b) = {}^{a'}[f(a)](b) \cdot [f(a')](b)$,

(5) $[f(a)]({}^{a'}b) = {}^{a'}[f(a)](b')$,

(6) $[f({}^{b'}a)]({}^{b'}b) = {}^{b'}[f(a)](b)$.

Proof. Define the maps $\alpha : \tilde{F}(A, Z^1(B, C)) \to \widetilde{\mathrm{Hom}}(A \otimes B, C)$, $\gamma : \widetilde{\mathrm{Hom}}(A \otimes B, C) \to \tilde{F}(A, Z^1(B, C))$ as follows:

$$[\alpha(f)](a \otimes b) = [f(a)](b), \quad f \in \tilde{F}(A, Z^1(B, C)), \ a \in A, b \in B,$$

$$[(\gamma(g))(a)](b) = g(a \otimes b), \quad g \in \widetilde{\mathrm{Hom}}(A \otimes B, C), \ a \in A, b \in B.$$

We shall show $[(\gamma(g))(a)] \in Z^1(B, C)$. We have $[(\gamma(g))(a)](bb') = g(a \otimes bb') = g(a \otimes b)g({}^{b}a \otimes {}^{b}b') = g(a \otimes b){}^{b}g(a \otimes b') = [(\gamma(g))(a)]\,(b) \cdot {}^{b}[(\gamma(g))(a)](b')$. Now we must show that $\gamma(g) \in \tilde{F}(A, Z^1(B, C))$. In fact, $[(\gamma(g))(aa')](b) = g(aa' \otimes b) = g({}^{a}a' \otimes {}^{a}b) \cdot g(a \otimes b) = {}^{a}g(a' \otimes b)g(a \otimes b) = {}^{a}[(\gamma(g))\ (a')](b) \cdot [(\gamma(g))(a)]\ (b) = g(a'a \otimes b) = {}^{a'}[(\gamma(g))(a)](b) \cdot [(\gamma(g))(a')](b)$. Furthermore, $[(\gamma(g))(a)]\ ({}^{a'}b) = g(a \otimes {}^{a'}b) = g({}^{a'}a \otimes {}^{a'}b) = {}^{a'}g(a \otimes b) = {}^{a'}[(\gamma(g))(a)](b)$, $[(\gamma(g))({}^{b'}a)]\ ({}^{b'}b) = g({}^{b'}a \otimes {}^{b'}b) = {}^{b'}g(a \otimes b) = {}^{b'}[(\gamma(g))(a)](b)$.

We shall prove that $\alpha(f) \in \widetilde{\mathrm{Hom}}(A \otimes B, C)$. For this consider the map $\overline{\alpha(f)}$: $A \times B \to C$, $[\overline{\alpha(f)}](a, b) = [f(a)](b)$. We have to show that this map is a crossed pairing. We have $[\overline{\alpha(f)}](aa', b) = [f(aa')](b) = {}^{a}[f(a')](b) \cdot [f(a)](b)$, $[\overline{\alpha(f)}]\ ({}^{a}a', {}^{a}b) = [f({}^{a}a')]({}^{a}b) = {}^{a}[f(a')](b)$. Thus, $[\overline{\alpha(f)}](aa', b) = [\overline{\alpha(f)}]({}^{a}a', {}^{a}b) \cdot [\overline{\alpha(f)}]\ (a, b)$. Next, $[\overline{\alpha(f)}](a,\ bb') = [f(a)](bb') = [f(a)](b) \cdot {}^{b}[f(a)](b')$, $[\overline{\alpha(f)}]({}^{b}a, {}^{b}b') = [f({}^{b}a)]({}^{b}b') = {}^{b}[f(a)](b')$. So, $[\overline{\alpha(f)}](a, bb') = [\overline{\alpha(f)}](a, b) \cdot [\overline{\alpha(f)}]({}^{b}a, {}^{b}b')$.

Therfore, the map $\overline{\alpha(f)}$ is a crossed pairing and it induces a homomorphism $\alpha(f) : A \otimes B \to C$. Now we prove that it satisfies conditions (1)–(3) of Theorem 2.

We have $[\alpha(f)](aa' \otimes b) = [f(aa')](b) = [f(a'a)](b) = [\alpha(f)] \quad (a'a \otimes b)$, $[\alpha(f)]({}^{a'}(a \otimes b)) = [\alpha(f)]({}^{a'}a \otimes {}^{a'}b) = [f({}^{a'}a)]({}^{a'}b) = {}^{a'}[f(a)](b) = {}^{a'}[\alpha(f)](a \otimes b)$, $[\alpha(f)]({}^{b'}(a \otimes b)) = [\alpha(f)]({}^{b'}a \otimes {}^{b'}b) = [f({}^{b'}a)]\,({}^{b'}b) = {}^{b'}[f(a)]\,(b) = {}^{b'}[\alpha(f)]\,(a \otimes b)$.
It is clear that $\alpha\gamma$ and $\gamma\alpha$ are identity maps. The proof is complete. □

In the case when the groups A, B and C are abelian and they act trivially, Theorem 2 is the well-known result that Hom and $\otimes$ are adjoint functors. The adjoining of these functors was extended to the category of precrossed modules (see [6, 10]).

We will show that Theorem 2 is a nontrivial generalization.

Example 1. A and B are abelian groups, C is an arbitrary group and the actions are trivial.

Example 2. Let $A = B$ be an arbitrary group, $C = A/H$, where H is the invariant subgroup of A generated by the elements $xyzy^{-1}x^{-1}\ yxz^{-1}x^{-1}y^{-1} (x, y, z \in A)$; the group A acts on C as follows: ${}^{a}[a'] = [aa'a^{-1}]$, $a, a' \in A$. Since in the group A/H holds the equality

$$xyzy^{-1}x^{-1}yxz^{-1}x^{-1}y^{-1} = e,$$

we have $x^{-1}y^{-1}xyz = zx^{-1}y^{-1}xy$ for $x, y, z \in A/H$. Thus, every commutator of A/H is central. Therefore, A/H is the maximal homomorphic metabelian image of A. Consider the homomorphisms $A \otimes A \xrightarrow{\gamma} A \xrightarrow{\tau} A/H$, where $\gamma(a \otimes a') = aa'a^{-1}a'^{-1}$ and τ is the canonical map. Then the homomorphism $g = \tau\gamma$, which in general is not trivial, satisfies conditions (1)–(3) of Theorem 2. In fact, $g(aa' \otimes a'') = [aa'a''a'^{-1}a^{-1}a''^{-1}] = [a'aa''a^{-1}a'^{-1}a''^{-1}] = g(a'a \otimes a'')$. Next, $g({}^{a'}a \quad \otimes {}^{a'}a'') = [{}^{a'}a{}^{a'}a''({}^{a'}a)^{-1}({}^{a'}a'')^{-1}] = [a'aa''a^{-1}a''^{-1}a'^{-1}] = {}^{a'}[aa''a^{-1}a''^{-1}] = {}^{a'}g(a \otimes a'')$.

Example 3. Let A be an abelian group, B an arbitrary group and the actions on each other be compatible. Put $C = A \otimes B$ and ${}^{a'}(a \otimes b) = {}^{a'}a \otimes {}^{a'}b$, ${}^{b'}(a \otimes b) = {}^{b'}a \otimes {}^{b'}b$. Then it is clear that the identity map $A \otimes B \to A \otimes B$ satisfies conditions (1)–(3).

Theorem 3. *Let $\{G_\alpha, \phi_\alpha^\beta, \alpha \le \beta\}$ be a directed system of groups. Let A be a group and let for every α the groups A, G_α act on each other and the homomorphisms ϕ_α^β preserve the actions. Then there is a natural isomorphism*

$$\left(\varinjlim_{\alpha} \{G_\alpha\}\right) \otimes A \approx \varinjlim_{\alpha} \{G_\alpha \otimes A\}.$$

Proof. Define the actions of $\varinjlim_{\alpha} \{G_\alpha\}$ and A on each other as follows: ${}^{[g_\alpha]}a = {}^{g_\alpha}a$, ${}^{a}[g_\alpha] = [{}^{a}g_\alpha]$ for all $g_\alpha \in G_\alpha$, $a \in A$. It is clear that these actions are correct. Next, define the map

$$\bar{\kappa}: \left(\varinjlim_{\alpha} \{G_\alpha\}\right) \times A \to \varinjlim_{\alpha} \{G_\alpha \otimes A\}$$

as follows: $\bar{\kappa}([g_\alpha], a) = [g_\alpha \otimes a]$. If $[g_\alpha] = [g_\beta]$, then there exists $\gamma \geq \alpha, \beta$ such that $\phi^\gamma_\alpha(g_\alpha) = \phi^\gamma_\beta(g_\beta)$. Thus, $(\phi^\gamma_\alpha)'(g_\alpha \otimes a) = \phi^\gamma_\alpha(g_\alpha) \otimes a = \phi^\gamma_\beta(g_\beta) \otimes a = (\phi^\gamma_\beta)'(g_\beta \otimes a)$, where $(\phi^\gamma_\alpha)'$, $(\phi^\gamma_\beta)'$ are induced, respectively, by ϕ^γ_α, ϕ^γ_β. Therefore, $[g_\alpha \otimes a] = [g_\beta \otimes a]$. We show that $\bar{\kappa}$ is a crossed pairing. We have $\bar{\kappa}([g_\alpha] \cdot [g_\beta], a) = [\phi^\gamma_\alpha(g_\alpha)\phi^\gamma_\beta(g_\beta) \otimes a]$, $\gamma \geq \alpha, \beta$; $\bar{\kappa}({}^{[g_\alpha]}[g_\beta], {}^{[g_\alpha]}a) = [\phi^\gamma_\alpha(g_\alpha)\phi^\gamma_\beta(g_\beta) \cdot \phi^\gamma_\alpha(g_\alpha^{-1}) \otimes {}^{g_\alpha}a]$; $\bar{\kappa}([g_\alpha], a) = [\phi^\gamma_\alpha(g_\alpha) \otimes a]$. Thus, $\bar{\kappa}([g_\alpha][g_\beta], a) = \bar{\kappa}({}^{[g_\alpha]}[g_\beta], {}^{[g_\alpha]}a)\bar{\kappa}([g_\alpha], a)$. Similarly, we can prove the second, the third and the fourth parts of the crossed pairing. Therefore, the map $\bar{\kappa}$ induces a homomorphism $\kappa : (\varinjlim_\alpha \{G_\alpha\}) \otimes A \to \varinjlim_\alpha \{G_\alpha \otimes A\}$. On the other hand, the canonical homomorphisms $\phi_\alpha : G_\alpha \otimes A \to (\varinjlim_\alpha \{G_\alpha\}) \otimes A$, $\phi_\alpha(g_\alpha \otimes a) = [g_\alpha] \otimes a$, induce a homomorphism $\kappa' : \varinjlim_\alpha \{G_\alpha \otimes A\} \to (\varinjlim_\alpha \{G_\alpha\}) \otimes A$ and it is easy to see that $\kappa\kappa'$, $\kappa'\kappa$ are identity maps. □

It would be preferable to formulate Theorem 3 in a more categorical way in terms of adjunction of functors amplying the preservation of colimits. In our general case the existence of the right adjoint functor for the nonabelian tensor product of groups is not known.

Now we consider the category $\mathscr{A}_A$ of groups acting on a fixed group A and the group A acts on these groups. Morphisms of the category $\mathscr{A}_A$ are homomorphisms of groups which preserve the actions. Let $F : \mathscr{A}_A \to \mathscr{A}_A$ be an endofunctor defined as follows: for $B \in \mathrm{ob}\,\mathscr{A}_A$ the object $F(B)$ is the free group generated by B with actions: ${}^{|b_1|^{\varepsilon_1} \cdots |b_s|^{\varepsilon_s}}a = {}^{b_1^{\varepsilon_1}}(\cdots({}^{b_s^{\varepsilon_s}}a)\cdots)$, and ${}^a(|b_1|^{\varepsilon_1} \cdots |b_s|^{\varepsilon_s}) = |{}^ab_1|^{\varepsilon_1} \cdots |{}^ab_s|^{\varepsilon_s}$ for $a \in A$, $|b_1|^{\varepsilon_1} \cdots |b_s|^{\varepsilon_s} \in F(B)$ and $\varepsilon_i = \pm 1$. If $f : B \to B'$ is a morphism of $\mathscr{A}_A$, then the canonical homomorphism $F(f)$ preserves the actions. We note that the actions of the groups $F(B)$ and A do not satisfy in general the compatibility conditions (1).

Let $\tau : F \to 1_{\mathscr{A}_A}$ be the obvious natural transformation and let $\delta : F \to F^2$ be the natural transformation induced for every $B \in \mathrm{ob}\,\mathscr{A}_A$ by the injection $B \to F(B)$. It is easy to verify that the homomorphisms τ_B and δ_B preserve the actions and therefore are morphisms of $\mathscr{A}_A$. We obtain a cotriple $\mathscr{F} = (F, \tau, \delta)$, which we call the free cotriple in the category $\mathscr{A}_A$.

If $\mathscr{P}$ is the projective class of objects in the category $\mathscr{A}_A$ induced by the free cotriple (F, τ, δ), we will show that any object $B \in \mathrm{ob}\,\mathscr{A}_A$ has a $\mathscr{P}$-projective pseudosimplicial resolution (for the definition see [7, 12]). Consider the canonical $\mathscr{P}$-projective resolution of B in the category of groups

$$\cdots \begin{smallmatrix}\longrightarrow \\ \vdots \\ \longrightarrow\end{smallmatrix} F(L_nB) \xrightarrow{\tau L_nB} L_nB \underset{l^n_n}{\overset{l^n_0}{\begin{smallmatrix}\longrightarrow \\ \vdots \\ \longrightarrow\end{smallmatrix}}} \cdots \begin{smallmatrix}\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow\end{smallmatrix} F(L_2B) \xrightarrow{\tau L_2B} L_2B \underset{l^2_2}{\overset{l^2_0}{\begin{smallmatrix}\longrightarrow \\ \longrightarrow \\ \longrightarrow\end{smallmatrix}}} F(L_1B)$$

$$\xrightarrow{\tau L_1B} L_1B \underset{l^1_1}{\overset{l^1_0}{\rightrightarrows}} F(B) \xrightarrow{\partial^0_0} B, \tag{4}$$

where $(L_{n+1}B, l_0^{n+1}, \ldots, l_{n+1}^{n+1})$, $n \geq 0$ is the simplicial kernel of the sequence of morphisms $(l_0^n \tau L_n B, \ldots, l_n^n \tau L_n B)$ (for the definition see [7, 12]). Each $F(L_n B)$ acts on A as follows: ${}^x a = {}^{\partial_0^0 \partial_0^1 \cdots \partial_0^n x} a$, where $\partial_0^0 = \tau_B$, $\partial_j^i = l_j^i \tau L_i B$, $a \in A$, $x \in F(L_n B)$. The action of A on $F(L_n B)$ will be defined by induction. The action of A on $F(B)$ induces the action of A on $L_1 B$ and therefore on $F(L_1 B)$. If the action of A is defined for all $F(L_i B)$, $1 \leq i \leq n-1$, then this induces the action of A on $L_n B$ and consequently on $F(L_n B)$. Note that for every $B \in \operatorname{ob} \mathscr{A}_A$ the morphism τ_B is $\mathscr{P}$-epimorphic. Thus, (4) is a $\mathscr{P}$-projective pseudosimplicial resolution of B in the category $\mathscr{A}_A$.

The nonabelian tensor product of groups defines a covariant functor $- \otimes A$ from the category $\mathscr{A}_A$ to the category $\mathfrak{Gr}$ of groups. Consider the nonabelian left-derived functors $L_n^{\mathscr{P}}(- \otimes A)$, $n \geq 0$, of the functor $- \otimes A$ relative to the projective class $\mathscr{P}$ [7]. It is known [11] that the derived functors relative to the projective class induced by a cotriple are isomorphic to the derived functors relative to this cotriple. Therefore, $L_n^{\mathscr{P}}(- \otimes A) \approx L_n^{\mathscr{F}}(- \otimes A)$, where the cotriple-derived functors $L_n^{\mathscr{F}}(B \otimes A)$ are defined using the cotriple resolution of B:

$$\cdots \ \begin{smallmatrix}\longrightarrow\\ \vdots \\ \longrightarrow\end{smallmatrix} \ F^n(B) \ \begin{smallmatrix}\longrightarrow\\ \vdots \\ \longrightarrow\end{smallmatrix} \ \cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow \\ \longrightarrow\end{smallmatrix} F^2(B) \rightrightarrows F^1(B) \longrightarrow B, \tag{5}$$

where $F^n(B) = F(F^{n-1}(B))$, $\partial_i^n = F^i \tau F^{n-i}$, $s_i^n = F^i \delta F^{n-i}$.

Theorem 4. *There is a natural isomorphism*

$$L_0^{\mathscr{P}}(- \otimes A) \approx - \otimes A.$$

Proof. Consider a projective pseudosimplicial resolution of an object B of the category $\mathscr{A}_A$:

$$\cdots \ \begin{smallmatrix}\longrightarrow\\ \vdots \\ \longrightarrow\end{smallmatrix} \ X_{n-1} \ \begin{smallmatrix}\xrightarrow{\partial_0^n}\\ \vdots \\ \xrightarrow[\partial_n^n]{}\end{smallmatrix} \ \cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow \\ \longrightarrow\end{smallmatrix} X_1 \begin{smallmatrix}\xrightarrow{\partial_0^1}\\ \xrightarrow[\partial_1^1]{}\end{smallmatrix} X_0 \xrightarrow{\partial_0^0} B.$$

Then the sequence of groups

$$\operatorname{Ker} \partial_0^1 \xrightarrow{\partial_1^1 \sigma} X_0 \xrightarrow{\partial_0^0} B \to 1$$

is exact, where $\sigma : \operatorname{Ker} \partial_0^1 \to X_1$ is the inclusion, and from Theorem 1 it follows that the sequence

$$\operatorname{Ker} \partial_0^1 \otimes A \xrightarrow{\partial_1^1 \sigma \otimes 1_A} X_0 \otimes A \xrightarrow{\partial_0^0 \otimes 1_A} B \otimes A \to 1$$

is also exact.

We have the commutative diagram

$$\begin{tikzcd}
\operatorname{Ker} \partial_0^1 \otimes A \arrow[r, "\partial_1^1 \sigma \otimes 1_A"] \arrow[d, "f"] & X_0 \otimes A \arrow[d, equal] & & \\
\operatorname{Ker}(\partial_0^1 \otimes 1_A) \arrow[r, "\partial_1^1 \sigma \otimes 1_A"] & X_0 \otimes A \arrow[r, "\partial_0^0 \otimes 1_A"] & B \otimes A \arrow[r] & 1,
\end{tikzcd}\ ;$$

where the homomorphism f induced by the inclusion σ is surjective by Theorem 1. Thus, the horizontal sequence of this diagram is exact.

Theorem 5. *For every short exact sequence of objects of the category* $\mathscr{A}_A$,

$$1 \longrightarrow B_1 \xrightarrow{f} B \xrightarrow{g} B_2 \longrightarrow 1,$$

we have the following exact sequences of the left-derived functors of the nonabelian tensor product:

$$\cdots \to L_{n+1}^{\mathscr{P}}(B_2 \otimes A) \to L_n^{\mathscr{P}}[(B, B_2, g) \otimes A] \to L_n^{\mathscr{P}}(B \otimes A) \to L_n^{\mathscr{P}}(B_2 \otimes A) \to \cdots, \tag{6}$$

where $L_n^{\mathscr{P}}[(B, B_2, g) \otimes A] = \pi_n(\mathrm{Ker}(F_*(g) \otimes \mathbf{1}_A)), n \geq 0,$

$$\mathrm{Ker}(F_*(g) \otimes \mathbf{1}_A) = \{\mathrm{Ker}(F^n(g) \otimes \mathbf{1}_A), n \geq 0\};$$

and, if A acts trivially on $(\mathrm{Ker}\, F_*(g) \to B_1)$,

$$\cdots \to L_{n+1}^{\mathscr{P}}[(B, B_2, g) \otimes A] \to L_n^{\mathscr{P}}[(B_1, B, f) \otimes A]$$
$$\to L_n^{\mathscr{P}}(B_1 \otimes A) \to L_n^{\mathscr{P}}[(B, B_2, g) \otimes A] \to \cdots, \tag{7}$$

where $L_n^{\mathscr{P}}[(B_1, B, f) \otimes A] = \pi_n(\mathrm{Ker}(F_*(f) \otimes \mathbf{1}_A)), n \geq 0,$

$$\mathrm{Ker}(F_*(f) \otimes \mathbf{1}_A) = \{\mathrm{Ker}(f_n \otimes \mathbf{1}_A), n \geq 0\},$$

(*see diagram* (8)).

Proof. The exact sequence $1 \to B_1 \xrightarrow{f} B \xrightarrow{g} B_2 \to 1$ induces the commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[d] & & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Ker}\, F^n(g) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_n"] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Ker}\, F^2(g) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_2"] & \mathrm{Ker}\, F^1(g) \arrow[r] \arrow[d, "f_1"] & B_1 \arrow[d, "f"] \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & F^n(B) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "F^n(g)"] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F^2(B) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "F^2(g)"] & F^1(B) \arrow[r] \arrow[d, "F^1(g)"] & B \arrow[d, "g"] \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & F^n(B_2) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F^2(B_2) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F^1(B_2) \arrow[r] \arrow[d] & B_2 \arrow[d] \\
 & 1 & & 1 & 1 & 1
\end{tikzcd} \tag{8}$$

where the bottom two rows are canonical cotriple resolutions, $\mathrm{Ker}(F_*(g))$ is a $\mathscr{P}$-projective resolution of B_1 if A acts trivially on $(\mathrm{Ker}\, F_*(g) \to B_1)$ and the homomorphisms f_n are inclusions.

If we take the tensor product of the diagram (8) and the group A we obtain the following commutative diagram:

$$
\begin{tikzcd}
\cdots \arrow[r, shift left, "\vdots"'] \arrow[r, shift right] & \mathrm{Ker}\, F^n(g) \otimes A \arrow[d, "f_n \otimes 1_A"] \arrow[r, shift left, "\vdots"'] \arrow[r, shift right] & \cdots \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Ker}\, F^1(g) \otimes A \arrow[r] \arrow[d, "f_1 \otimes 1_A"] & B_1 \otimes A \arrow[d, "f \otimes 1_A"] \\
\cdots \arrow[r, shift left, "\vdots"'] \arrow[r, shift right] & \mathrm{Ker}\, (F^n(g) \otimes 1_A) \arrow[d, tail] \arrow[r, shift left, "\vdots"'] \arrow[r, shift right] & \cdots \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Ker}\, (F^1(g) \otimes 1_A) \arrow[r] \arrow[d, tail] & \mathrm{Ker}\, (g \otimes 1_A) \arrow[d, tail] \\
\cdots \arrow[r, shift left, "\vdots"'] \arrow[r, shift right] & F^n(B) \otimes A \arrow[d, "F^n(g) \otimes 1_A"] \arrow[r, shift left, "\vdots"'] \arrow[r, shift right] & \cdots \arrow[r, shift left] \arrow[r, shift right] & F^1(B) \otimes A \arrow[r] \arrow[d, "F^1(g) \otimes 1_A"] & B \otimes A \arrow[d, "g \otimes 1_A"] \\
\cdots \arrow[r, shift left, "\vdots"'] \arrow[r, shift right] & F^n(B_2) \otimes A \arrow[r, shift left, "\vdots"'] \arrow[r, shift right] & \cdots \arrow[r, shift left] \arrow[r, shift right] & F^1(B_2) \otimes A \arrow[r] & B_2 \otimes A
\end{tikzcd}
$$

From Theorem 2 it follows that all vertical homomorphisms of this diagram, except the natural inclusions, are surjective. One gets two short exact sequences of pseudosimplicial groups and the assertion follows from Theorem 1.3 [7]. $\square$

Theorem 6. *If A is an abelian group which acts trivially on a group $G \in \mathrm{ob}\, \mathscr{A}_A$, then we have natural isomorphisms*

$$L_n^{\mathscr{P}}(G \otimes A) \approx H_{n+1}(G, A), \quad n \geq 1,$$

$$\mathrm{Ker}\, \lambda' \approx H_1(G, A), \quad \mathrm{Coker}\, \lambda' \approx H_0(G, A),$$

where $\lambda' : G \otimes A \to A$, $\lambda'(g \otimes a) = {}^g a \cdot a^{-1}$.

Proof. Let A be a G-module, let $\mathfrak{Gr}_G$ be the category of groups over G and let $\mathrm{Diff}_G(W) = \boldsymbol{Z}[G] \otimes_W IW$ for W a group over G. By Guin's Proposition 3.2 of [6], $L_n^{\mathscr{P}}(- \otimes A)$ is isomorphic to the nth left-derived functor of $A \otimes_{Z[G]} \mathrm{Diff}_G(-) : \mathfrak{Gr}_G \to \mathfrak{Gr}$ which gives the Eilenberg–Maclane homology group $H_{n+1}(G, A)$ if $n \geq 1$ (see [1]). The proof is complete. $\square$

Definition 2. Let G be an object of the category $\mathscr{A}_A$. Then we define

$$H_n(G, A) = L_{n-1}^{\mathscr{P}}(G \otimes A), \quad n \geq 2, \qquad H_1(G, A) = \mathrm{Ker}\, \lambda', \qquad H_0(G, A) = \mathrm{Coker}\, \lambda',$$

where $\lambda' : G \otimes A \to A/H'$, $\lambda'(g \otimes a) = [{}^g a a^{-1}]$, and H' is the normal subgroup generated by the elements ${}^{({}^a g)} a'^{\, a g a^{-1}} a'^{-1}$, for each $a, a' \in A$, $g \in G$.

Remark 2. It is clear that the groups $H_n(G, A)$ are abelian for $n \geq 2$. We will show that $\operatorname{Im}\lambda'$ is a normal subgroup of A/H' and when for the actions of G and A the compatibility conditions (1) hold, then $H_1(G, A)$ is also abelian.

Suppose $[a] \in \operatorname{Im}\lambda'$ and $[b] \in A/H'$. We have $[b][a][b]^{-1} = [b]\lambda'[(g_1 \otimes a_1) \cdots (g_n \otimes a_n)][b^{-1}] = [b^{g_1}a_1a_1^{-1} \cdots {}^{g_n}a_na_n^{-1}b^{-1}] = [{}^{bg_1b^{-1}}({}^ba_1)({}^ba_1)^{-1} \cdots {}^{bg_nb^{-1}}({}^ba_n)({}^ba_n)^{-1}] = [{}^{({}^bg_1)}({}^ba_1)][({}^ba_1)^{-1}] \cdots [{}^{({}^bg_n)}({}^ba_n)][({}^ba_n)^{-1}] = \lambda'[({}^bg_1 \otimes {}^ba_1) \cdots ({}^bg_n \otimes {}^ba_n)]$, where $\lambda'[(g_1 \otimes a_1) \cdots (g_n \otimes a_n)] = [a]$. It follows $[b][a][b]^{-1} \in \operatorname{Im}\lambda'$ and $\operatorname{Im}\lambda'$ is a normal subgroup of A.

Take $l \in \operatorname{Ker}\lambda'$. From Proposition 2.3(f) of [4] we have $(g \otimes a)l = (g \otimes a)l \times (g \otimes a)^{-1}l^{-1}l(g \otimes a) = [(g \otimes a), l]l(g \otimes a) = (\lambda(g \otimes a) \otimes \lambda'(l))l(g \otimes a) = l(g \otimes a)$, where $\lambda: G \otimes A \to G$, $\lambda(g \otimes a) = g^a g^{-1}$, $\lambda': G \otimes A \to A$, $\lambda'(g \otimes a) = {}^g a a^{-1}$ because $H' = 1$. It follows that $\operatorname{Ker}\lambda' \subset \mathscr{Z}(G \otimes A)$ and therefore it is an abelian group.

Definition 3. If A is a G-group, then we define the homology of G with coefficients in A as $H_*(G, A)$ with trivial action of A on G.

Theorem 7. *Let G, A_1, A, A_2 be arbitrary groups, G acts on A_1, A and A_2, which act on G. Let $1 \to A_1 \xrightarrow{f} A \xrightarrow{g} A_2 \to 1$ be an exact sequence of groups, where f and g homomorphisms preserve the actions. Then there exist exact sequences of the nonabelian homology*

$$\cdots \to H_3(G, A_2) \to H_2(G, A, A_2) \to H_2(G, A) \to H_2(G, A_2)$$

$$\xrightarrow{l_1} H_1(G, A, A_2) \xrightarrow{l_2} H_1(G, A) \to H_1(G, A_2) \xrightarrow{l_3} H_0(G, A, A_2)$$

$$\xrightarrow{l_4} H_0(G, A) \longrightarrow H_0(G, A_2) \longrightarrow 1, \tag{9}$$

where

$$H_n(G, A, A_2) = \pi_{n-1}(\operatorname{Ker}(1_{F_*(G)} \otimes g)), n \geq 2,$$

$$\operatorname{Ker}(1_{F_*(G)} \otimes g) = \{\operatorname{Ker}(1_{F^n(G)} \otimes g), n \geq 1\},$$

$$H_1(G, A, A_2) = \frac{[\operatorname{Ker}(1_{F^1(G)} \otimes g) \cap \partial_0^{0^{-1}}(\operatorname{Ker}(1_G \otimes g) \cap \operatorname{Ker}\lambda')]}{\partial_1^1(\operatorname{Ker}(1_{F^2(G)} \otimes g) \cap \operatorname{Ker}\partial_0^1)}$$

$$H_0(G, A, A_2) = \operatorname{Ker}\tilde{g}/\lambda'(\operatorname{Ker}(1_G \otimes g)),$$

and

$$\cdots \to H_3(G, A, A_2) \to H_2(G, A_1, A) \to H_2(G, A_1) \to H_2(G, A, A_2)$$

$$\to H_1(G, A_1, A) \to H_1(G, A_1) \to H_1(G, A, A_2)$$

$$\to H_0(G, A_1, A) \to H_0(G, A_1) \to H_0(G, A, A_2) \to 1, \tag{10}$$

where the groups $H_n(G, A_1, A)$ are defined analogously.

Proof. Follows from the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F^2(G)\otimes A_1 \arrow[r, shift left, "\partial_0^1"] \arrow[r, shift right, "\partial_1^1"'] \arrow[d, "1_{F^2(G)}\otimes f"] & F^1(G)\otimes A_1 \arrow[r, "\partial_0^0"] \arrow[d, "1_{F^1(G)}\otimes f"] & G\otimes A_1 \arrow[r, "\lambda'"] \arrow[d, "1_G\otimes f"] & A_1/H'_1 \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Ker}\,(1_{F^2(G)}\otimes g) \arrow[r, shift left, "\bar{\partial}_0^1"] \arrow[r, shift right, "\bar{\partial}_1^1"'] \arrow[d] & \mathrm{Ker}\,(1_{F^1(G)}\otimes g) \arrow[r, "\bar{\partial}_0^0"] \arrow[d] & \mathrm{Ker}\,(1_G\otimes g) \arrow[r, "\bar{\lambda}'"] \arrow[d] & \mathrm{Ker}\,\tilde{g} \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F^2(G)\otimes A \arrow[r, shift left, "\partial_0^1"] \arrow[r, shift right, "\partial_1^1"'] \arrow[d, "1_{F^2(G)}\otimes f"] & F^1(G)\otimes A \arrow[r, "\partial_0^0"] \arrow[d, "1_{F^1(G)}\otimes f"] & G\otimes A \arrow[r, "\lambda'"] \arrow[d, "1_G\otimes f"] & A/H' \arrow[d, "\tilde{g}"] \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F^2(G)\otimes A_2 \arrow[r, shift left, "\partial_0^1"] \arrow[r, shift right, "\partial_1^1"'] & F^1(G)\otimes A_2 \arrow[r, "\partial_0^0"] & G\otimes A_2 \arrow[r, "\lambda'"] & A_2/H_2.
\end{tikzcd}$$ □

Remark 3. The sequence (9) generalizes the well-known classical exact sequence of the homology of groups with abelian coefficients. If A_1, A, A_2 are G-modules, then the groups $H_n(G, A_1, A)$ are trivial. The exact sequence of the nonabelian homology $H_*(G, A)$ of groups with respect to the variable G can be obtained in the same way by using Theorem 5.

Remark 4. (a) If G and A act on each other compatibly (in this case G, A_1 and G, A_2 act on each other compatibly), then $H_0(G, A, A_2) = H_0(G, A_1)$.

(b) Let $1 \to (A_1, 1) \to (A, \mu) \to (A_2, \lambda) \to 1$ be an exact sequence of crossed G-modules. Then Guin has obtained [6] the following exact sequence of the non-abelian homology:

$$\begin{aligned} &H_1(G, A_1) \to H_1(G, A) \to H_1(G, A_2) \to H_0(G, A_1) \\ &\to H_0(G, A) \to H_0(G, A_2) \to 1. \end{aligned} \tag{11}$$

The first five terms of the sequence (9) coincide with the sequence (11) and we have a natural homomorphism $H_1(G, A_1) \to H_1(G, A, A_2)$.

Let A be a noncommutative local ring such that $A/\mathrm{Rad}\,A \neq \mathbb{F}_2$. From [6, 9] we know that there exists a group $U(A)$ generated by elements $\langle u, v\rangle$, where $u, v \in A^*$ (A^* is the group of units of A), subject to the relations

(U0) $\langle u, 1-u\rangle = 1$, $u \neq 1$, $u, 1-u \in A^*$,

(U1) $\langle uu', v\rangle = {}^u\langle u', v\rangle\langle u, v\rangle$,

(U2) $\langle u, vw\rangle\langle v, wu\rangle\langle w, uv\rangle = 1$,

where ${}^u\langle v, w\rangle = \langle uvu^{-1}, uwu^{-1}\rangle$, such that we have a short exact sequence of crossed A^*-modules

$$1 \to K_2(A) \to U(A) \to [A^*, A^*] \to 1,$$

$$\langle u, v\rangle \mapsto [u, v],$$

and we have the following exact sequence of nonabelian homology:

$$H_1(A^*, K_2(A)) \to H_1(A^*, U(A)) \to H_1(A^*, [A^*, A^*]) \to H_0(A^*, K_2(A))$$
$$\to H_0(A^*, U(A)) \to H_0(A^*, [A^*, A^*]) \to 1. \tag{12}$$

It can be proved [6] that $H_0(A^*, [A^*, A^*]) = [A^*, A^*]/[A^*, [A^*, A^*]]$; $H_0(A^*, U(A)) = \mathrm{Sym}(A)$; $H_0(A^*, K_2(A)) = K_2(A)$, where the group $\mathrm{Sym}(A)$ is generated by elements $\{u, v\}$, where $u, v \in A^*$, subject to the relations

(S1) $\{u, 1-u\} = 1, u \neq 1, u, 1-u \in A^*$,
(S2) $\{uu', v\} = \{u, v\}\{u', v\}$,
(S3) $\{u, vv'\} = \{u, v\}\{u, v'\}$.

Corollary. *Let A be a noncommutative local ring such that $A/\mathrm{Rad}(A) \neq \mathbb{F}_2$. Then the following sequence of groups is exact:*

$$\cdots \to H_3(A^*, [A^*, A^*]) \to H_2(A^*, U(A), [A^*, A^*]) \to H_2(A, U(A))$$
$$\to H_2(A^*, [A^*, A^*]) \to H_1(A^*, U(A), [A^*, A^*]) \to H_1(A, U(A))$$
$$\to H_1(A^*, [A^*, A^*]) \to K_2(A) \to \mathrm{Sym}(A) \to [A^*, A^*]/[A^*, [A^*, A^*]] \to 1.$$

Now we compute H_0 and H_1 for some nonabelian groups. It is clear that $H_0(G, G) = {}^{ab}G$, where G is arbitrary group.

Since the commutator map $\kappa: G \otimes G \to G$ is the universal central extension (covering group) of the perfect group G (see [2], Corollary 1), it follows that $H_1(G, G) \approx H_2(G, \boldsymbol{Z})$.

Consider the well-known short exact sequence of groups

$$0 \to K_2(A) \to \mathrm{ST}(A) \to E(A) \to e,$$

where $E(A)$, $ST(A)$ and $K_2(A)$ are the elementary group, the Steinberg group and the Milnor K-group of the ring A with unit, respectively. This sequence induces the actions of these groups on each other by conjugation. Since $K_2(A) \otimes E(A)$ is a trivial group, the induced map $ST(A) \otimes E(A) \to E(A) \otimes E(A)$ is an isomorphism. Hence, $H_1(ST(A), E(A)) \approx K_2(A)$. It can also be proved that $H_0(ST(A), E(A)) = H_0(E(A), ST(A)) = H_1(E(A), ST(A)) = 1$.

Theorem 8. *Let $\{G_\alpha, \Phi_\alpha^\beta, \alpha \leq \beta\}$ and $\{A_\alpha, \Psi_\alpha^\beta, \alpha \leq \beta\}$ be directed systems of groups. Let G and A be groups and let for every α the groups A, G_α and G, A_α act on each other and the homomorphisms Φ_α^β, Ψ_α^β preserve the actions. Then we have natural isomorphisms*

$$H_n\left(\varinjlim_\alpha \{G_\alpha\}, A\right) \approx \varinjlim_\alpha \{H_n(G_\alpha, A)\}, \quad n \geq 0,$$

$$H_n\left(G, \varinjlim_\alpha \{A_\alpha\}\right) \approx \varinjlim_\alpha \{H_n(G, A_\alpha)\}, \quad n \geq 0.$$

Proof. First we prove

$$H_0\left(\varinjlim_\alpha \{G_\alpha\}, A\right) \approx \varinjlim_\alpha \{H_0(G_\alpha, A)\},$$

$$H_0\left(G, \varinjlim_\alpha \{A_\alpha\}\right) \approx \varinjlim_\alpha \{H_0(G, A_\alpha)\}.$$

Define the maps $f: H_0(G, \varinjlim_\alpha \{A_\alpha\}) \to \varinjlim_\alpha \{H_0(G, A_\alpha)\}$, $g: \varinjlim_\alpha \{H_0(G, A_\alpha)\} \to H_0(G, \varinjlim_\alpha \{A_\alpha\})$ and $f': H_0(\varinjlim_\alpha \{G_\alpha\}, A) \to \varinjlim_\alpha \{H_0(G_\alpha, A)\}$, $g': \varinjlim_\alpha \{H_0(G, A_\alpha)\} \to H_0(G, \varinjlim_\alpha \{A_\alpha\})$ as follows: $f([\{a_\alpha\}]) = \{[a_\alpha]\}$, $\{a_\alpha\} \in \varinjlim_\alpha \{A_\alpha\}$; the well defined homomorphisms $g_\alpha: H_0(G, A_\alpha) \to H_0(G, \varinjlim_\alpha \{A_\alpha\})$, $[a_\alpha] \mapsto [\{a_\alpha\}]$, induce the homomorphism g; $f'([a]) = \{[a]\}, g'\{[a]\} = [a]$, $a \in A$. It is easy to see that these homomorphisms are well defined and fg, gf, $f'g'$ and $g'f'$ are identity maps.

Now we prove

$$H_1\left(\varinjlim_\alpha \{G_\alpha\}, A\right) \approx \varinjlim_\alpha \{H_1(G_\alpha, A)\},$$

$$H_1\left(G, \varinjlim_\alpha \{A_\alpha\}\right) \approx \varinjlim_\alpha \{H_1(G, A_\alpha)\}.$$

Define the maps $f: H_1(G, \varinjlim_\alpha \{A_\alpha\}) \to \varinjlim_\alpha \{H_1(G, A_\alpha)\}$, $g: \varinjlim_\alpha \{H_1(G, A_\alpha)\} \to H_1(G, \varinjlim_\alpha \{A_\alpha\})$ and $f': H_1(\varinjlim_\alpha \{G_\alpha\}, A) \to \varinjlim_\alpha \{H_1(G_\alpha, A)\}$, $g': \varinjlim_\alpha \{H_1(G, A_\alpha)\} \to H_1(G, \varinjlim_\alpha \{A_\alpha\})$ as follows: the natural isomorphisms $G \otimes \varinjlim_\alpha \{A_\alpha\} \approx \varinjlim_\alpha \{G \otimes A_\alpha\}$ and $\varinjlim_\alpha \{G_\alpha\} \otimes A \approx \varinjlim_\alpha \{G_\alpha \otimes A\}$ (see Theorem 3) induce f, g and f', g' respectively. It is clear that fg, gf and $f'g'$, $g'f'$ are identity maps.

Finally, we prove $H_n(G, \varinjlim_\alpha \{A_\alpha\}) \approx \varinjlim_\alpha \{H_n(G, A_\alpha)\}$, for $n \geq 2$, $H_n(\varinjlim_\alpha \{G_\alpha\}, A) \approx \varinjlim_\alpha \{H_n(G_\alpha, A)\}$, for $n \geq 2$. Applying the well-known assertions $\pi_n(\varinjlim_\alpha \{D_\alpha\}) \approx \varinjlim_\alpha \{\pi_n(D_\alpha)\}$, where D_α is a simplicial group and $F(\varinjlim_\alpha \{G_\alpha\}) \approx \varinjlim_\alpha \{F(G_\alpha)\}$, we obtain the required isomorphisms $H_n(G, \varinjlim_\alpha \{A_\alpha\}) \approx \varinjlim_\alpha \{H_n(G, A_\alpha)\}$, $n \geq 2$, and $H_n(\varinjlim_\alpha \{G_\alpha\}, A) \approx \varinjlim_\alpha \{H_n(G_\alpha, A)\}$, $n \geq 2$. $\square$

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